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When a Random Walk of Fixed Length can Lead Uniformly Anywhere Inside a Hypersphere

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A variation of the Pearson-Rayleigh random walk in which the steps are i.i.d. random vectors of exponential length and uniform orientation is considered. Conditioned on the total path length, the probability density function of the position of the walker after n steps is determined analytically in one and two dimensions. It is shown that in two dimensions $n = 3$ marks a critical transition point in the behavior of the random walk. By taking less than three steps and walking a total length l, one is more likely to end the walk near the boundary of the disc of radius l, while by taking more than three steps one is more likely to end near the origin. Somehow surprisingly, by taking exactly three steps one can end uniformly anywhere inside the disc of radius l. This means that conditioned on l, the sum of three vectors of exponential length and uniform direction has a uniform probability density.

While the presented analytic approach provides a complete solution for all n, it becomes intractable in higher dimensions. In this case, it is shown that a necessary condition to have a uniform density in dimension d is that $2(d + 2)/d$ is an integer, equal to $n + 1$.

KEY WORDS: random walks, Pearson-Rayleigh walk, brownian motion, applied probability

1. INTRODUCTION

"A man starts from a point O and walks a yards in a straight line; he then turns through any angle whatever and walks another a yards in a second straight line. He repeats this process n times. I require the probability that after n of these stretches he is at distance between r and $r + \delta r$ from his starting point O." Karl Pearson, 1905.

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Above description defines the Pearson-Rayleigh random walk on the continuum plane. In this paper we consider the variant in which the step lengths are i.i.d. exponential random variables. Imagine a particle moving in a random environment and undergoing elastic collisions at uniformly distributed point obstacles. Accordingly, the steps of the particle are i.i.d. random vectors of exponential length and uniform orientation. We determine the probability density function of the distance from the starting point after n steps, conditioned on the total travelled length l. A formal solution is easily constructed using characteristic functions, and evaluation in closed form is made by analytic manipulations and by exploiting known properties of Bessel functions.

In the two-dimensional case, we show that after three steps the walk can end uniformly anywhere inside a ball of radius l. For $n < 3$ the walk ends at a point that is likely to be near the boundary of the ball of radius l, while for $n \to \infty$ it tends to localize at the origin. This confirms the intuition that in one step the walk consists of a single straight line of lenght l, but as the number of steps increases and the total path length is kept fixed to l, single steps become on average smaller, and the walk entangles around the origin. Surprisingly, the case $n = 3$ marks a critical transition point when the density function is uniformly spread and does accumulate neither at the origin, nor at the boundary of the area spanned by the walk. A similar transition point is observed in one dimension after two, rather than three steps. In this case, the conditional distribution has half the mass distributed uniformly in the ball of radius l, and a delta mass of 1/4 at each end point of the interval. On the other hand, the transition point is absent in three and all dimensions for which $2(d + 2)/d$ is not an integer and equal to $n + 1$.

It is easy to show the one-dimensional case. Condition on the sum of the Euclidean lengths of the steps to be a constant $l > 0$. Starting from the origin choose a positive direction with probability 1/2, and note that the first exponential step ends at coordinate X_1, uniformly chosen in the interval $[0, l]$. The second step either follows the same direction and ends at distance l from the origin (with probability 1/4), or it follows the opposite direction and ends at random coordinate $X_2 = X_1 - (l - X_1) = 2X_1 - l$. Since X_1 is uniform in $[0, l]$, it immediately follows that the density of X_2 is also uniform in the interval $[-l, l]$ and has two Dirac's delta functions $\frac{1}{4}\delta(x + l)$, $\frac{1}{4}\delta(x - l)$ placed at the end points of the interval. Clearly, the presence of the delta functions is due to the possibility of the two steps being taken in the same direction, which does not occur in higher dimensions. The simple argument given, however, does not generalize to dimensions two and above.

The next section illustrates the complete one dimensional solution. Section 3 illustrates the two dimensional case. Section 4 shows that the uniform density does not arise in all dimensions for which $2(d + 2)/d$ is not an integer and equal to $n + 1$. Finally, we want to point out the book by Hughes,[(1)] which provides a detailed background on the topic of random walks.

2. ONE DIMENSION

Starting from the origin, the walker chooses a random direction and takes a first step of exponential length. Accordingly, let the probability density function (pdf) of reaching random coordinate $X \in \mathbb{R}$ in the first step on the line be given by

$$f_X(x) = \frac{\eta}{2} e^{-\eta|x|}. \tag{1}$$

Note that since the direction is isotropic, above density depends only on the absolute value $|x|$ and that it integrates to one over the real line. Consider now the joint pdf of reaching coordinate X on the line in a step of Euclidean length $L \in \mathbb{R}^+$. It is easy to write this using Dirac's $\delta(\cdot)$ function,

$$f_{X,L}(x,l) = \frac{\eta}{2} e^{-\eta|x|} \delta(l - |x|). \tag{2}$$

We now iterate. The position after n steps and the total path length travelled are given by the sum of n i.i.d. random vectors. Hence, their joint pdf can be computed by performing $m = n - 1$ convolution operations of the single steps,

$$f_{\sum_{i=1}^{n}(X_i,L_i)}(x,l) = \overbrace{f_{X,L} * f_{X,L} * \ldots * f_{X,L}(x,l)}^{m} \equiv f_{m_{X,L}}(x,l). \tag{3}$$

Letting $\Phi_{X,L}(\omega, \chi)$ be the Fourier transform (i.e. characteristic function) of $f_{X,L}(x,l)$, we can write the equivalent of (3) in the spectral domain as

$$\Phi_{m_{X,L}}(\omega, \chi) = (\Phi_{X,L}(\omega, \chi))^{m+1}, \tag{4}$$

where

$$\begin{aligned}
\Phi_{X,L}(\omega, \chi) &= \int_{-\infty}^{\infty} dx \int_0^{\infty} dl e^{-i\omega x} e^{-i\chi l} f_{X,L}(x,l) \\
&= \frac{\eta}{2} \int_{-\infty}^{\infty} e^{-i\omega x} e^{-\eta|x|} dx \int_0^{\infty} e^{-i\chi l} \delta(l - |x|)\, dl \\
&= \frac{\eta}{2} \int_{-\infty}^{\infty} e^{-(\eta + i\chi)|x|} e^{-i\omega x} dx \\
&= \frac{\eta}{2} \int_0^{\infty} e^{-\eta x} e^{-i(\omega+\chi)x} dx + \frac{\eta}{2} \int_{-\infty}^{0} e^{\eta x} e^{-i(\omega-\chi)x} dx \\
&= \eta \frac{\eta + i\chi}{(\eta + i\chi)^2 + \omega^2}.
\end{aligned} \tag{5}$$

We now raise (5) to the $m + 1$ power and then compute the inverse transform. Since the solution is symmetric around the origin, to simplify the notation we can restrict our attention to the interval $x > 0$. Substituting (5) into (4) and writing the inverse transform we obtain,

$$f_{m_{X,L}}(x,l) = \frac{1}{(2\pi)^2}\eta^{m+1}\int_{-\infty}^{\infty} e^{i\omega x}d\omega \int_{-\infty}^{\infty} e^{i\chi l}\frac{(\eta+i\chi)^{m+1}}{[(\eta+i\chi)^2+\omega^2]^{m+1}}d\chi. \quad (6)$$

Let us write the integral in ω as follows,

$$\int_{-\infty}^{\infty}\frac{e^{i\omega x}}{[(\eta+i\chi)^2+\omega^2]^{m+1}}d\omega = 2\int_0^{\infty}\frac{\cos\omega x}{[(\eta+i\chi)^2+\omega^2]^{m+1}}d\omega$$
$$= 2\sqrt{\pi}\frac{x^{m+\frac{1}{2}}}{2^{m+\frac{1}{2}}}\frac{1}{(\eta+i\chi)^{m+\frac{1}{2}}}\frac{K_{m+\frac{1}{2}}[(\eta+i\chi)x]}{\Gamma(m+1)}, \quad (7)$$

where the last equality follows from identity 8.432.5 of Ref. 2, and $K_m(\cdot)$ is the modified Bessel function of the first kind. Let us now proceed to compute the remaining integral. We use the following polynomial expansion of the function $K_m(\cdot)$, see identity 8.468 of Ref. 2,

$$K_{m+\frac{1}{2}}[(\eta+i\chi)x] = \sqrt{\frac{\pi}{2}\frac{1}{(\eta+i\chi)x}}e^{-(\eta+i\chi)x}\sum_{k=0}^{m}\frac{(m+k)!}{2^k k!(m-k)!}\frac{1}{[(\eta+i\chi)x]^k} \quad (8)$$

which substituted into (7) and then substituting into (6) leads to

$$f_{m_{X,L}}(x,l) = \frac{1}{(2\pi)^2}\eta^{m+1}\frac{\pi}{m!2^m}e^{-\eta x}\sum_{k=0}^{m}\frac{(m+k)!}{k!(m-k)!}\frac{x^{m-k}}{2^k}\int_{-\infty}^{\infty}\frac{e^{i(l-x)\chi}}{(\eta+i\chi)^k}d\chi. \quad (9)$$

For $k=0$ the integral in (9) reduces to

$$\int_{-\infty}^{\infty} e^{i(l-x)\chi}d\chi = 2\pi\delta(l-x). \quad (10)$$

For $k>0$ and letting $p=i\chi$, the integral in (9) can be computed as a known inverse Laplace transform of a function differentiated with respect to η,

$$\int_{-\infty}^{\infty}\frac{e^{i(l-x)\chi}}{(\eta+i\chi)^k}d\chi = \frac{(-1)^{k-1}}{(k-1)!}\frac{\partial^{k-1}}{\partial\eta^{k-1}}\int_{-\infty}^{\infty}\frac{e^{i(l-x)\chi}}{(\eta+i\chi)}d\chi$$
$$= \frac{(-1)^{k-1}}{(k-1)!}\frac{\partial^{k-1}}{\partial\eta^{k-1}}\frac{1}{i}\int_{-i\infty}^{i\infty}\frac{e^{p(l-x)}}{(\eta+p)}dp$$
$$= \frac{2\pi}{(k-1)!}(l-x)^{k-1}e^{-\eta(l-x)}, \quad (l>x). \quad (11)$$

Substituting (10) and (11) into (9) we obtain,

$$f_{m_{X,L}}(x,l) = \frac{\eta^{m+1}}{(2\pi)^2}\frac{\pi}{m!2^m}e^{-\eta x}\left(f^{(1)}(x,l)+f^{(2)}(x,l)\right) \quad (12)$$

where

$$f^{(1)}(x,l) = 2\pi e^{-\eta(l-x)} \sum_{k=1}^{m} \frac{(m+k)!}{k!(m-k)!} \frac{x^{m-k}}{(k-1)!} \frac{(l-x)^{k-1}}{2^k}, \quad (l > x)$$

$$f^{(2)}(x,l) = 2\pi \delta(l-x) x^m. \tag{13}$$

It is easy to check that for $m = 0$ (12–13) reduce to (2), the exponential pdf of a single step of length l. Finally, we can compute the conditional density after n steps. First, we notice that the pdf of the total path length travelled in n steps, obtained by m-fold convolution of

$$f_L(l) = \eta e^{-\eta l}, \quad (l > 0), \tag{14}$$

is simply the Gamma density

$$f_{m_L}(l) = \frac{\eta^{m+1}}{m!} l^m e^{-\eta l}, \quad (l > 0), \tag{15}$$

which together with (12) leads to,

$$\begin{aligned} f_{m_{X|L}}(x,l) &= \frac{f_{m_{X,L}}(x,l)}{f_{m_L}(l)} \\ &= \frac{1}{(2\pi)^2} \frac{\pi}{(2l)^m} e^{-\eta(x-l)} \left(f^{(1)}(x,l) + f^{(2)}(x,l) \right). \end{aligned} \tag{16}$$

Above result holds for $x > 0$ and is symmetric for $x < 0$, so it is now easy to check that for $m = 1$ the conditional density of the two-step random walk is the linear combination of a uniform density and two Dirac's delta functions placed at the boundary of the interval $[-l, l]$, as anticipated in the introduction. Obtained formulas are valid for any number of steps, but do not give insight on what happens in higher dimensions.

3. TWO DIMENSIONS

It is possible to obtain a closed form solution in two dimensions using the same analytic approach outlined above. Again, formulas hold for any number of steps. In the special case of three steps, they reveal that, conditioned on the sum of their absolute values, the vectorial sum of three random vectors of exponential length and uniform direction has a uniform density. This means that being constrained by a path of fixed length l, by taking three random steps on the plane the walker can end uniformly anywhere inside the ball of radius l.

Consider first the total path length travelled on the plane (i.e. the sum of the Euclidean lengths of the single steps of the walk). As in the one dimensional case, this is given by $m = n - 1$ convolutions of the one dimensional, positive, exponential density. Hence, we rewrite the Gamma density

$$f_{m_L}(l) = \frac{\eta^{m+1}}{m!} l^m e^{-\eta l}, (l > 0). \tag{17}$$

We now consider the first step of the random walk on the plane. Starting from the origin, the walker chooses a random direction uniformly in $[0, 2\pi]$ and takes a step of exponential length. Accordingly, the pdf of the random coordinate $R \in \mathbb{R}^2$ reached in the first step is given by

$$f_R(\mathbf{r}) = \frac{\eta}{2\pi r} e^{-\eta r}. \tag{18}$$

Note that above density depends only on the absolute value $r = |\mathbf{r}|$ and that it integrates to one over the whole plane. Consider now the joint pdf of reaching coordinate R on the plane in a single step of length $L \in \mathbb{R}^+$. This is immediately given by

$$f_{R,L}(\mathbf{r}, l) = \frac{\eta}{2\pi r} e^{-\eta r} \delta(l - r). \tag{19}$$

Proceeding as in the one dimensional case, we now compute the joint pdf after n steps by performing $m = n - 1$ convolutions of (19), and then by exploiting (17) we obtain the conditional pdf

$$f_{m_{R|L}}(\mathbf{r}, l) = \frac{f_{m_{R,L}}(\mathbf{r}, l)}{f_{m_L}(l)}. \tag{20}$$

We start by computing the Fourier transform of (19). In cartesian coordinates this is written as

$$\Phi_{R,L}(u, v, \chi) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_0^{\infty} dl \frac{\eta}{2\pi\sqrt{x^2 + y^2}} e^{-\eta\sqrt{x^2+y^2}} \\ \times \delta\left(l - \sqrt{x^2 - y^2}\right) e^{-i(ux+vy)} e^{-i\chi l}. \tag{21}$$

By letting $x = r\cos\theta$, $y = r\sin\theta$, $u = \omega\cos\psi$, $v = \omega\sin\psi$, we have

$$\Phi_{R,L}(\omega, \psi, \chi) = \int_0^{2\pi} d\theta \int_0^{\infty} r dr \int_0^{\infty} dl \frac{\eta}{2\pi r} e^{-\eta r} \delta(l - r) e^{-i\omega r\cos(\theta-\psi)} e^{-i\chi l}. \tag{22}$$

The θ integration is performed by expanding the complex exponential in Bessel functions $J_k(\cdot)$ using identity 8.511.4 of Ref. 2, obtaining

$$\Phi_{R,L}(\omega, \psi, \chi) = \frac{\eta}{2\pi} \int_0^{\infty} e^{-\eta r} dr \int_0^{\infty} e^{-i\chi l} \delta(l - r)\, dl \\ \times \int_0^{2\pi} \sum_{k=-\infty}^{\infty} (-i)^k J_k(\omega r) e^{-ik(\theta-\psi)} d\theta$$

$$= \eta \int_0^\infty J_0(\omega r) e^{-\eta r} dr \int_0^\infty e^{-i\chi l} \delta(l - r)\, dl$$

$$= \eta \int_0^\infty J_0(\omega r) e^{-(\eta + i\chi) r} dr$$

$$= \frac{\eta}{\sqrt{(\eta + i\chi)^2 + \omega^2}}, \tag{23}$$

where the last equality follows from identity 6.611.1 of Ref. 2. Note that (23) depends only on ω and χ and not on ψ, as expected, as the transformed pdf depends only on r and l and not on θ.

We now perform the convolutions in the spectral domain and compute the inverse Fourier transform. The integral in ψ of the inverse transform is obtained following exactly the same procedure as in (23), which leads to

$$f_{m_{R,L}}(\mathbf{r}, l) = \frac{\eta^{m+1}}{(2\pi)^2} \int_{-\infty}^\infty e^{i\chi l} d\chi \int_0^\infty \frac{\omega J_0(\omega r)}{[\omega^2 + (\eta + i\chi)^2]^{\frac{m+1}{2}}} d\omega. \tag{24}$$

We compute the integral in ω by exploiting identity 6.565.4 of Ref. 2, obtaining

$$\int_0^\infty \frac{\omega J_0(\omega r)}{[\omega^2 + (\eta + i\chi)^2]^{\frac{m+1}{2}}} d\omega = \frac{r^{\frac{m-1}{2}}}{2^{\frac{m-1}{2}} \Gamma\left(\frac{m+1}{2}\right)} \frac{K_{-\frac{m-1}{2}}[(\eta + i\chi) r]}{(\eta + i\chi)^{\frac{m-1}{2}}}. \tag{25}$$

For $m > 0$, renaming $i\chi + \eta = p$, and taking into account that $K_{-m}(\cdot) = K_m(\cdot)$, we can identify the last integral in χ with the Laplace transform 2.13.21 of Ref. 3 that yields

$$\frac{1}{2\pi} \int_{-\infty}^\infty \frac{K_{-\frac{m-1}{2}}[(i\chi + \eta) r]}{(i\chi + \eta)^{\frac{m-1}{2}}} e^{i\chi l} d\chi = e^{-\eta l} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{K_{\frac{m-1}{2}}(pr)}{p^{\frac{m-1}{2}}} e^{pl} dp$$

$$= e^{-\eta l} \frac{\sqrt{\pi} (l^2 - r^2)^{\frac{m-2}{2}}}{2^{\frac{m-1}{2}} r^{\frac{m-1}{2}} \Gamma\left(\frac{m}{2}\right)}, \quad (l > r). \tag{26}$$

Combining Eqs. (19) and (24)–(26), we finally get

$$\begin{cases} f_{0_{R,L}}(\mathbf{r}, l) = \dfrac{\eta}{2\pi r} e^{-\eta r} \delta(l - r) \\[2ex] f_{m_{R,L}}(\mathbf{r}, l) = \dfrac{\eta}{2\pi} e^{-\eta l} \dfrac{2\sqrt{\pi} \eta^m}{2^m \Gamma(\frac{m+1}{2}) \Gamma(\frac{m}{2})} (l^2 - r^2)^{\frac{m-2}{2}}, \quad (l > r), \ m = 1, 2, \ldots \end{cases} \tag{27}$$

Equation (27) is a closed form expression for the joint pdf of reaching position $\mathbf{r}$ in $n = m + 1$ steps, and with total path length l. The conditional probability is readily obtained by substituting (17) and (27) into (20) and exploiting the *doubling formula* of the Gamma function, see formula 8.335.1 of Ref. 2. After some algebra,

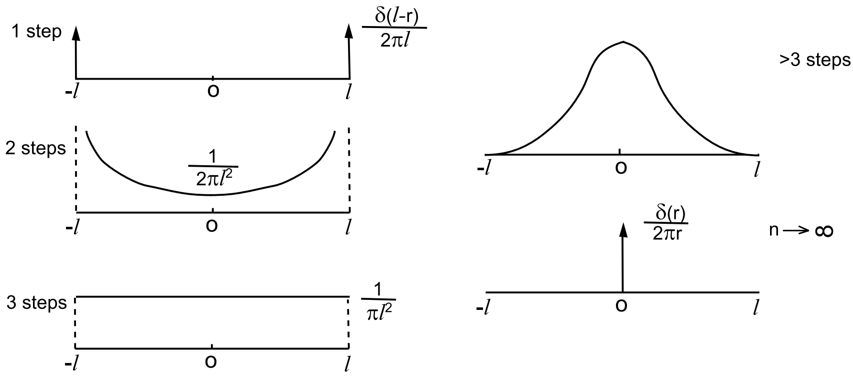

Fig. 1. We sketch the probability density function of the position of the walker after n steps, conditioned on the total path length. Note that taking three steps of the walk on the plane, marks a critical transition point between the probability mass being concentrated on the boundaries and being concentrated at the origin of the walk. At this transition point the random walk can lead uniformly anywhere inside the disc of radius l.

one obtains the simple expression

$$\begin{cases} f_{0_{R|L}}(\mathbf{r}, l) = \dfrac{1}{2\pi l}\delta(l-r) \\ f_{m_{R|L}}(\mathbf{r}, l) = \dfrac{m}{2\pi l^2}\left(1 - \dfrac{r^2}{l^2}\right)^{\frac{m-2}{2}}, \quad (l > r), \quad m = 1, 2, \ldots \end{cases} \tag{28}$$

It is now interesting to look at some special cases that are also sketched in Fig. 1.

- For $m = 0$ (one step),

$$f_{0_{R|L}}(\mathbf{r}, l) = \frac{1}{2\pi l}\delta(l-r),$$

the whole probability mass is concentrated on the circle of radius l. Clearly, in one step a random walk constrained by a total path length l can only end at a position uniformly distributed on this circle.

- For $m = 1$ (two steps),

$$f_{1_{R|L}}(\mathbf{r}, l) = \left(1 - \frac{r^2}{l^2}\right)^{-\frac{1}{2}} \frac{1}{2\pi l^2},$$

the mass is still mostly concentrated around the circle of radius l. However, taking two steps allows some flexibility on where to end the walk and there is some non-zero mass distributed inside the disc of radius l.

- For $m = 2$ (three steps),

$$f_{2_{R|L}}(\mathbf{r}, l) = \frac{1}{\pi l^2},$$

the mass is now uniformly distributed over the disc of radius l. Hence, taking three steps of the walk marks a critical transition point where the random walk can end uniformly anywhere inside the disc.

- For $m \to \infty$

$$f_{\infty_{R|L}}(\mathbf{r}, l) = \frac{\delta(r)}{2\pi r},$$

the mass tends to be concentrated at the origin. Clearly, by taking a large number of steps and fixing the total path length, uniformly oriented steps on average must become smaller and this tends to entangle the walk around the origin point.

4. THREE AND HIGHER DIMENSIONS

Computations become intractable in higher dimensions. Hence, rather than seeking for a complete solution, we turn to the question of whether the uniform density arises at all for a given number of steps in dimension d. A simple second moment argument shows that a necessary condition for this is that $2(d + 2)/d$ is an integer and equal to $n + 1$, which clearly rules out the three dimensional case. The computation is briefly outlined next.

We let the total path length be $L = \sum_{i=1}^{n} L_i$, where L_i's are one dimensional i.i.d. exponential random variables of density $f_{L_i}(l) = \eta e^{-\eta l}$, $(l > 0)$. The second moment of L is given by,

$$\begin{aligned} E(L^2) &= \sum_{i=1}^{n} E\left(L_i^2\right) + E\left(\sum_{i \neq j} L_i L_j\right) \\ &= \frac{2n}{\eta^2} + \frac{n(n-1)}{\eta^2}, \\ &= \frac{n(n+1)}{\eta^2}, \end{aligned} \tag{29}$$

where we have used the independence of the L_i's and that their mean is $1/\eta$ and their variance is $1/\eta^2$. We also write the second moment of the final position R of the random walk via the conditional expectation,

$$E(|R|^2) = E(E(|R|^2|L)). \tag{30}$$

Recall now that for a random vector Z uniformly distributed in a hypersphere of radius s in d dimensions it must be,

$$E(|Z|^2) = \frac{d}{d+2}\, s^2. \tag{31}$$

It follows from (30) and (31) that if R conditioned to the total path length L is uniformly distributed it must be,

$$E(|R|^2) = E(E(|R|^2|L)) = \frac{d}{d+2}E(L^2). \tag{32}$$

Hence, substituting (29) into (32) we have that a necessary condition for the uniform density to arise is,

$$E(|R|^2) = \frac{d}{d+2}\frac{n(n+1)}{\eta^2}. \tag{33}$$

On the other hand, we can write R in terms of its d-dimensional components,

$$R = \left(\sum_{i=1}^{n} R_{i1}, \sum_{i=1}^{n} R_{i2}, \ldots, \sum_{i=1}^{n} R_{id}\right), \tag{34}$$

and using the fact that each d-dimensional component has zero mean, the second moment can be easily computed as follows

$$\begin{aligned} E(|R|^2) &= \sum_{j=1}^{d}\sum_{i=1}^{n} E\left(R_{ij}^2\right) + \sum_{i_1 \neq i_2} E(R_{i_1 j})E(R_{i_2 j}) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{d} E\left(R_{ij}^2\right) = \sum_{i=1}^{n} E\left(L_i^2\right) = \frac{2n}{\eta^2}, \end{aligned} \tag{35}$$

Combining (33) and (35) we have following necessary condition for the conditional density after n steps to be uniform,

$$n = \frac{2(d+2)}{d} - 1. \tag{36}$$

We conclude that in three and all dimensions for which $2(d+2)/d$ is not an integer and equal to $n+1$, the random walk does not lead to a uniform density. Furthermore, we notice that the necessary condition is clearly not sufficient for $d = 1$.

5. CONCLUSION

We have considered a variant of the Pearson-Rayleigh random walk where the lengths of the steps are i.i.d. exponentially distributed random variables. We have considered the pdf of the position of the walker after n steps, conditioned on the total length l of the path and obtained closed form solutions in one and two dimensions. We have noticed that in two dimensions by taking three steps of the walk, one can end uniformly anywhere inside a ball of radius l. Since it is trivial to show that a similar situation arises by taking only two steps in one dimension, one might naturally suspect to obtain a uniform density after four steps in three dimensions. On the contrary, we have shown that this is not the case, and that in $d-$dimensions a necessary condition to obtain a uniform density is $2(d+1)/d = n+1$.

We mention that after being surprised to find the uniform density arising after three steps on the plane, we have tried to come up with a simple geometric argument for this case, but did not succeed. It is likely that a geometric proof alternative to the analytic calculation may give additional insight on the behavior of the walk in higher dimensions, and remains an open problem. Furthermore, it would be interesting to see if performing higher moment computations it is possible to obtain a sufficient condition for uniformity in dimension d.

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